

# Constructing Series

Anton 11.10

Important Maclaurin Series: memorize

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$(-\infty, \infty)$

What is the interval of convergence?

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \left| \frac{x}{n+1} \right| \rightarrow 0 < 1 \Rightarrow \sum \frac{x^n}{n!} \text{ conv. for all } x,$$

→

Important Maclaurin Series: memorize

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$(-\infty, \infty)$



What is the interval of convergence?

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}}}{\frac{(2n+3)(2n+2)}{(2n+1)(2n+2)}} \right| = \left| \frac{x^2}{(2n+3)(2n+2)} \right| \rightarrow 0 < 1$$

Important Maclaurin Series: memorize

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$(-\infty, \infty)$



What is the interval of convergence?



### Important Maclaurin Series: memorize

$$1 + x + x^2 + x^3 + \dots + x^n + \dots = \underbrace{\frac{1}{1-x}}_{\substack{\text{VALID IF SERIES} \\ \text{CONVERGES}}} = \sum_{n=0}^{\infty} x^n$$

What is the interval of convergence?

Geom. series conv. if  $|r| < 1$

$$\begin{array}{c} |x| < 1 \\ -1 < x < 1 \end{array}$$



Use substitution with a known series to find the series expansion of:

$$\frac{1}{1+x}$$

use  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + \dots$

$$\frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \dots + (-x)^n + \dots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots = \sum_{n=0}^{\infty} (-1)^n x^n$$

INT OF CONV:  $(-1, 1)$



Use substitution with a known series to find the series expansion of:

$$e^{-x} \Rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots + \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots - \frac{(-1)^n (x^n)}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

INT OF CONV:  $(-\infty, \infty)$



Use substitution with a known series to find the series expansion of:

$$\frac{1}{1-2x^2} \Rightarrow \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-(2x^2)} = 1 + (2x^2) + (2x^2)^2 + (2x^2)^3 + \cdots + (2x^2)^n + \cdots = \sum_{n=0}^{\infty} (2x^2)^n$$

$$= 1 + 2x^2 + 2^2 x^4 + 2^3 x^6 + \cdots + 2^n x^{2n} + \cdots = \sum_{n=0}^{\infty} 2^n x^{2n}$$

$|2x^2| < 1$   
 $-1 < 2x^2 < 1$   
 $0 < x^2 < 1/2$   
 $|x| < \sqrt{1/2} \Rightarrow \boxed{-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}}$



## Differentiation and Integration of Series:

$$\begin{aligned}
 \frac{d}{dx}(\sin x) &= \frac{d}{dx} \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots - \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots \right) \\
 &= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \dots - \frac{(-1)^n (2n+1)x^{2n}}{(2n+1)!} + \dots \\
 &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots - \frac{(-1)^n x^{2n}}{(2n)!} + \dots \\
 &= \cos x
 \end{aligned}$$

→

## Differentiation and Integration of Series:

$$\begin{aligned}
 \int e^x dx &= \int \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \right) dx \\
 &= x + \frac{x^2}{2} + \frac{x^3}{3 \cdot 2!} + \frac{x^4}{4 \cdot 3!} + \dots + \frac{x^{n+1}}{(n+1) \cdot n!} + \dots + C \\
 &= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^{n+1}}{(n+1)!} + \dots + C \quad \text{turns into } 1. \\
 &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{n+1}}{(n+1)!} + \dots \\
 &= e^x
 \end{aligned}$$

→

Use differentiation or integration of a known series to find the series expansion for:

$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x} dx \\ &= \int (1-x+x^2-x^3+x^4-\dots+(-1)^n x^n+\dots) dx \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^n x^{n+1}}{(n+1)} + \dots + C \\ \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots + \frac{(-1)^n x^{n+1}}{(n+1)} + \dots\end{aligned}$$



Use differentiation or integration of a known series to find the series expansion for:

$$\begin{aligned}\tan^{-1} x &= \int \frac{1}{1+x^2} dx \\ &= \int (1-x^2+x^4-x^6+\dots+(-1)^n x^{2n}+\dots) dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots + C \\ \tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + \boxed{(-1)^n \frac{x^{2n+1}}{2n+1}}\end{aligned}$$



Use any method to find the series expansion of:

$$\begin{aligned} f(x) = x^3 \sin x &= x^3 \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots \right) \\ &= x^4 - \frac{x^6}{3!} + \frac{x^8}{5!} - \frac{x^{10}}{7!} + \dots (-1)^n \frac{x^{2n+4}}{(2n+1)!} + \dots \end{aligned}$$



Use any method to find the series expansion of:

$$\begin{aligned} f(x) = \frac{e^{-x^2}}{x} \Rightarrow e^{-x^2} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \\ e^{-x^2} &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots (-1)^n \frac{x^{2n}}{n!} + \dots \\ \frac{e^{-x^2}}{x} &= \frac{1}{x} - x + \frac{x^3}{2!} - \frac{x^5}{3!} + \dots (-1)^n \frac{x^{2n-1}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{n!} \end{aligned}$$



Use any method to find the series expansion of:

$$f(x) = \frac{2x^2}{1 - 3x^3}$$



Important Series:

See page 572 in Anton; See page 477 in FDWK



**Homework:**

**Anton 11.10 #27 - 41 odd**



